

## ALGEBRAIC CURVES EXERCISE SHEET 2

### Exercise 2.1.

- (1) Let  $V$  be an algebraic set in  $\mathbf{A}^n(k)$  and  $P \in \mathbf{A}^n(k)$  a point not in  $V$ . Show that there is a polynomial  $F \in k[X_1, \dots, X_n]$  such that  $F(Q) = 0$  for all  $Q \in V$ , but  $F(P) = 1$ .
- (2) Let  $P_1, \dots, P_r$  be distinct points in  $\mathbf{A}^n(k)$ , not in an algebraic set  $V$ . Show that there are polynomials  $F_1, \dots, F_r \in I(V)$  such that  $F_i(P_j) = 0$  if  $i \neq j$ , and  $F_i(P_i) = 1$ .
- (3) With  $P_1, \dots, P_r$  and  $V$  as in (2), and  $a_{ij} \in k$  for  $1 \leq i, j \leq r$ , show that there are  $G_i \in I(V)$  with  $G_i(P_j) = a_{ij}$  for all  $i$  and  $j$ .

### Exercise 2.2.

- (1) Determine which of the following sets are algebraic:
  - (a)  $\{(x, y) \in \mathbf{A}^2(\mathbf{R}) \mid y = \sin(x)\}$
  - (b)  $\{(\cos(t), \sin(t)) \in \mathbf{A}^2(\mathbf{R}) \mid t \in \mathbf{R}\}$
  - (c)  $\{(z, w) \in \mathbf{A}^2(\mathbf{C}) \mid |z|^2 + |w|^2 = 1\}$
- (2) Show that any algebraic subset of  $\mathbf{A}^n(\mathbf{R})$  can be defined by a single polynomial equation. Is the same true for  $\mathbf{A}^n(\mathbf{C})$ ?

### Exercise 2.3.

Let  $k$  be a field and  $I, J$  two ideals of  $k[x_1, \dots, x_n]$ . Let  $a = (a_1, \dots, a_n) \in k^n$ . Recall that  $I \cdot J = \{fg, f \in I, g \in J\}$ . Show the following assertions:

- (1) If  $I \subseteq J$ , then  $V(J) \subseteq V(I)$ .
- (2)  $V(I) \cup V(J) = V(I \cdot J)$ .
- (3)  $V(\{x_1 - a_1, \dots, x_n - a_n\}) = \{a\}$ .

### Exercise 2.4.

Let  $V \subseteq \mathbb{A}_k^n$  and  $W \subseteq \mathbb{A}_k^m$  be algebraic sets. Show that the following set is an algebraic subset of  $\mathbb{A}_k^{m+n}$ :

$$V \times W = \{(a_1, \dots, a_n, b_1, \dots, b_m) \in \mathbb{A}_k^{m+n} \mid (a_1, \dots, a_n) \in V, (b_1, \dots, b_m) \in W\}$$

**Exercise 2.5.**

A ring is called *local* if it has a unique maximal ideal. Let  $k$  be an algebraically closed field.

- (1) Let  $I \subseteq R = k[x_1, \dots, x_n]$  be an ideal such that  $V(I)$  is a point. Show that  $R/I$  is a finite-dimensional local algebra and that the elements of the maximal ideal are nilpotent.
- (2) Let  $I \subseteq R = k[x_1, \dots, x_n]$  be a *radical* ideal such that  $V(I)$  is a finite set of  $r$  points. Show that  $R/I \simeq k \times \dots \times k$ , with  $r$  copies of  $k$  in the product. (Hint: consider the intersection of maximal ideals containing  $I$  and use the chinese remainder theorem).

**Exercise 2.6.**

Let  $k$  be an algebraically closed field and  $V = \{p_1, \dots, p_r\} \subseteq \mathbb{A}_k^n$  a finite algebraic set. We call  $a_i$ ,  $1 \leq i \leq s$  the distinct first coordinates of  $p_1, \dots, p_r$ . Consider the finite varieties  $V_i = \{(x_2, \dots, x_n) \in \mathbb{A}^{n-1} \mid (a_i, x_2, \dots, x_n) \in V\} \subseteq \mathbb{A}^{n-1}$ .

- (1) Assume that each  $V_i$  is the zero locus of  $N$  polynomials  $f_{i,1}, \dots, f_{i,N}$  for some  $N \geq 1$ . Show that there exist polynomials  $g_k$ ,  $1 \leq k \leq N$  such that  $g_k(a_i, x_2, \dots, x_n) = f_{i,k}$ .
- (2) Show that  $V$  is the zero locus of  $n$  polynomials in  $k[x_1, \dots, x_n]$ . (Hint: reason by induction on  $n$ )
- (3) Show that  $I(V)$  is generated by  $n$  polynomials. (Hint: using the previous exercise,  $I(V)$  is characterized by  $k[x_1, \dots, x_n]/I(V) \simeq k \times \dots \times k$ )